$\boxed{\text{HW\#2 Solutions}}$

1) Superposition

a) $x(t) = \cos\omega t + \sin(\omega t + \pi/4) \rightarrow z(t) = e^{i\omega t} - i e^{i\omega t + i\pi/4} = e^{i\omega t}\left[1 - i e^{i\pi/4}\right] = e^{i\omega t}\left[1 + e^{i(\pi/4 - \pi/2)}\right] = e^{i\omega t}\left[\underbrace{1 + e^{-i\pi/4}}\right]$

$1 + \cos\pi/4 - i\sin\pi/4 = 1 + \frac{1}{\sqrt{2}} - i/\sqrt{2} = re^{i\alpha}$

with $r = \left[(1 + 1/\sqrt{2})^2 + (1/\sqrt{2})^2\right]^{1/2} = \left[2 + \sqrt{2}\right]^{1/2} = 1.85$

$\alpha = \arctan\left(\frac{-1/\sqrt{2}}{1 + 1/\sqrt{2}}\right) = -.392$

$$\boxed{z = 1.85\, e^{i(\omega t - .392)}}$$

b) $x(t) = \cos(\omega t + \pi/3) + \cos(\omega t) \rightarrow z(t) = e^{i(\omega t + \pi/3)} + e^{i\omega t} = e^{i\omega t}\left[1 + e^{i\pi/3}\right] = e^{i\omega t}\left[1 + \cos\pi/3 + i\sin\pi/3\right] = e^{i\omega t}\left[\underbrace{\tfrac{3}{2} + i\tfrac{\sqrt{3}}{2}}\right]$

$re^{i\alpha}$ with $r = \left[(\tfrac{3}{2})^2 + (\tfrac{\sqrt{3}}{2})^2\right]^{1/2} = \sqrt{6}$

$\alpha = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \pi/6$

$$\boxed{z(t) = \sqrt{6}\, e^{i(\omega t + \pi/6)}}$$

c) $x(t) = 2\cos\omega t + 3\sin\omega t \rightarrow z(t) = 2e^{i\omega t} - 3i e^{i\omega t} = e^{i\omega t}\left[\underbrace{2 - 3i}\right]$

$re^{i\alpha}$ with $r = \left[2^2 + 3^2\right]^{1/2} = \sqrt{13}$ $\quad \alpha = \arctan(-3/2) = -.98$

$$\boxed{z(t) = \sqrt{13}\, e^{i(\omega t - .98)}}$$

d) $x(t) = \sin\omega t + \cos\omega t - 2\cos(\omega t + \pi/4) \rightarrow z(t) = -i e^{i\omega t} + e^{i\omega t} - 2e^{i(\omega t + \pi/4)} = e^{i\omega t}\left[1 - i - 2\cos\pi/4 - 2i\sin\pi/4\right] = e^{i\omega t}\left[\underbrace{1 - \sqrt{2} - i(1 + \sqrt{2})}\right]$

$re^{i\alpha}$ with $r = \left[(1 - \sqrt{2})^2 + (1 + \sqrt{2})^2\right]^{1/2} = \sqrt{6}$

$\alpha = \arctan\left(-\frac{(1 + \sqrt{2})}{(1 - \sqrt{2})}\right) = -1.74$

$$\boxed{z(t) = \sqrt{6}\, e^{i(\omega t - 1.74)}}$$

2) Periodicity of superposed oscillations

in general, want arguments of both functions to be multiples of 2π: for $\left.\begin{array}{l} x_1 = A_1\cos(r\omega t) \\ x_2 = A_2\cos(s\omega t) \end{array}\right\}$ $\begin{array}{l} r\omega T = 2\pi M \\ s\omega T = 2\pi N \end{array}$ ↕ integers

Solve for T: $\quad T = \frac{2\pi M}{r\omega} = \frac{2\pi N}{s\omega}$, or $\quad \frac{M}{N} = \frac{r}{s}$ $\quad$ goal is then to find reduced fraction, if possible, to satisfy equality

a) $x_1 = A_1\cos\omega t \quad x_2 = A_2\cos(2\omega t)$: $\quad \frac{M}{N} = \frac{1}{2} \Rightarrow M = 1, N = 2 \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$

b) $\frac{M}{N} = \frac{r}{s} = \frac{2}{3}$ $\quad M = 2,\ N = 3,\ \boxed{T = \frac{4\pi}{\omega}}$

c) $\frac{M}{N} = \frac{m}{n} \rightarrow$ not obvious what exact ans. is here, but can say $\frac{m}{n} = \frac{q}{p}$ ← red. fraction and $M = q,\ N = p$ and $\boxed{T = \frac{2\pi q}{n\omega} = \frac{2\pi p}{m\omega}}$

d) $\frac{M}{N} = \frac{\pi}{1} \rightarrow$ NOT possible to express as ratio of integers $\Rightarrow$ $\boxed{\text{NO PERIODICITY} \quad T = \infty}$

3) $x_1 = A\cos(\omega_1 t)$ $x_2 = A\cos\omega_2 t$ $z(t) = Ae^{i\omega_1 t} + Ae^{i\omega_2 t}$

$$\left.\begin{array}{l}\bar\omega = \frac{1}{2}(\omega_1+\omega_2)\\ \Delta\omega = \omega_2-\omega_1\end{array}\right\} \begin{array}{l}\omega_1 = \bar\omega - \frac{1}{2}\Delta\omega\\ \omega_2 = \bar\omega + \frac{1}{2}\Delta\omega\end{array}$$

$$\Rightarrow z = A\left[e^{i(\bar\omega - \frac{1}{2}\Delta\omega)t} + e^{i(\bar\omega + \frac{1}{2}\Delta\omega)t}\right]$$
$$= A e^{i\bar\omega t}\left[e^{\frac{i}{2}\Delta\omega t} + e^{-i\frac{\Delta\omega}{2}t}\right]$$
$$= A e^{i\bar\omega t}\cos\left(\tfrac{1}{2}\Delta\omega t\right)$$

Take real part: $\boxed{x(t) = Re[z(t)] = Re\left[Ae^{i\bar\omega t}\cos(\tfrac{1}{2}\Delta\omega t)\right] = A\cos(\bar\omega t)\cos(\tfrac{1}{2}\Delta\omega t)}$

4) a) $\omega = \sqrt{\frac{k}{m}}$, $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$

b) With two springs, $F = -(2k)x = m\ddot{x} \Rightarrow \ddot{x} = -\omega^2 x = -\left(\frac{2k}{m}\right)x \Rightarrow \omega = \sqrt{\frac{2k}{m}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{2k}} = \sqrt{2}\,\pi\sqrt{\frac{m}{k}}}$

c) For total displacement x, each spring stretches by $\frac{x}{2}$, so $F = -k\frac{x}{2} = m\ddot{x}$, or $\ddot{x} = -\frac{k}{2m}x \Rightarrow \omega = \sqrt{\frac{k}{2m}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{2m}{k}} = 2\sqrt{2}\,\pi\sqrt{\frac{m}{k}}}$