HW#1 Solutions

① a) If the bottom of the range of motion is 10cm below, than the amplitude of oscillation is 5cm, and thus the equilibrium position is 5cm below the point where the spring would be at rest without the mass. Using Hooke's Law, and balancing the force due to gravity, we have $F_g = -F_{spring}$, or $Mg = k\Delta x$ with $\Delta x = 5cm$. This allows us to solve for $\omega^2 = \frac{k}{M} = \frac{g}{\Delta x}$

$$\boxed{\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{g}{\Delta x}} = \sqrt{\frac{980\ cm/s^2}{5cm}} = 14\ rad/s}$$

b) We can use conservation of energy here. At 5cm below initial spot, $U = 0$, and K.E. $\left(\frac{1}{2}mv^2\right)$ is maximum. At resting place, $U = \frac{1}{2}kx^2$ is max, with $x = 5cm$

Conserving energy, $U_{max} = K.E._{max}$, or $\frac{1}{2}kx_{max}^2 = \frac{1}{2}mv_{max}^2$, or $\boxed{v_{max} = x_{max}\sqrt{\frac{k}{m}} = x_{max}\,\omega = (5cm)(14\,rad/s) = 70\,cm/s}$

c) $\boxed{\omega_{new} = \sqrt{\frac{k}{M+300gm}} = \frac{1}{2}\omega_{old} = \frac{1}{2}\sqrt{\frac{k}{M}},\ \text{or}\ M = 100gm}$

d) The response of the spring to mass is linear, and $\boxed{M_{new} = 4M \Rightarrow \Delta X_{new} = 4\Delta X_{old} = 4(5cm) = 20cm}$

② a) $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$ and $\cos(-\theta) = \cos(\theta)$, $\sin(-\theta) = -\sin\theta$ (sin and cos are "odd" and "even" functions, respectively)

so: $\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

b) Note $\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) = \frac{1}{2}(2\cos\theta) = \cos\theta \rightarrow \boxed{\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})}$

c) Note $\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(\cos\theta + i\sin\theta - \cos\theta + i\sin\theta) = \frac{1}{2i}(2i\sin\theta) = \sin\theta \rightarrow \boxed{\sin\theta = \frac{1}{2i}(e^{i\theta} + e^{-i\theta})}$

d) $\cos(a\pm b) = Re[e^{i(a\pm b)}] = Re[e^{ia}e^{\pm ib}] = Re[(\cos a + i\sin a)(\cos(\pm b) + i\sin(\pm b))] = Re[(\cos a + i\sin a)(\cos b \pm i\sin b)] = Re[\cos a\cos b \mp \sin a\sin b + i(\sin a\cos b \pm \cos a\sin b)]$

$= \cos a\cos b \mp \sin a\sin b$

so $\boxed{\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b}$

e) $\boxed{\sin(a\pm b) = Im[e^{i(a\pm b)}] = \sin a\cos b \pm \cos a\sin b}$ Note: We have used this intermediate step of last part!

③ $\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$ (by prop. of exp. and chain rule) $\frac{d}{d\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta)$ (by properties of trig. functions!)

④ $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots = \sum_{\substack{n=0 \\ (n\text{ even})}}^{\infty} (-1)^{\frac{n}{2}} \frac{\theta^n}{n!} = \sum_{\substack{n=0 \\ (n\text{ even})}} \frac{i^n \theta^n}{n!} = \sum_{\substack{n=0 \\ (n\text{ even})}}^{\infty} \frac{(i\theta)^n}{n!}$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\theta^n}{n!} = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} (i)^{n-1} \frac{\theta^n}{n!} = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{1}{i} \frac{(i\theta)^n}{n!}$

And $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$ $\quad e^{-i\theta} = \sum_{n=0}^{\infty} \frac{(-i\theta)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{(i\theta)^n}{n!}$

even terms same

odd terms opposite sign!

So: $\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}\left(\sum_{\substack{n=0 \\ (n\text{ even})}}^{\infty} 2\frac{(i\theta)^n}{n!}\right) = \sum_{\substack{n=0 \\ (n\text{ even})}}^{\infty} \frac{(i\theta)^n}{n!} = \cos\theta$ (by comparison above)

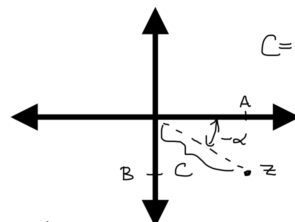
And: $\frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}\left(\sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} 2\frac{(i\theta)^n}{n!}\right) = \sum_{\substack{n=1 \\ (n\text{ odd})}}^{\infty} \frac{1}{i}\frac{(i\theta)^n}{n!} = \sin\theta$ (by comparison above)

⑤ We plug in proposed solution + check that LHS=RHS: $\frac{d^2}{dt^2}(A\cos\omega t + B\sin\omega t) = (A(-\omega^2)\cos\omega t + B(-\omega^2)\sin(\omega t)) = -\omega^2(A\cos\omega t + B\sin\omega t) = -\omega^2 y(t)$

Now use complex exponentials: $A\cos\omega t + B\sin\omega t = \mathrm{Re}[Ae^{i\omega t} - iBe^{i\omega t}] = \mathrm{Re}[(A - iB)e^{i\omega t}] = \mathrm{Re}[(Ce^{i\alpha})e^{i\omega t}]$

complex # $z = A - iB$ can be written as $Ce^{i\alpha}$:

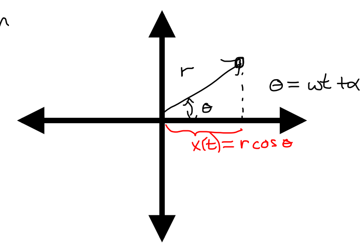
$C = \sqrt{A^2 + B^2} \quad \alpha = -\arctan\left(\frac{B}{A}\right)$



③ a) $\omega = \frac{2\pi}{T} = \frac{2\pi}{(6\sec)} = 1.05$ rad/s $\quad \alpha = \pi/6$, and $v = \omega r = 100$ cm/s $\Rightarrow r = \frac{100\text{ cm/s}}{1.05\text{ rad/s}} = 95$ cm

*I will assume motion is CCW, even though it was not specified.

Flip sign of ω for CW motion!



$x(t) = (95\text{cm})\cos[(1.05\,\tfrac{\text{rad}}{\text{s}})t + \pi/6]$

b) $x(2\sec) = (95\text{cm})\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) = -82\text{cm}$

$\dot{x}(2\sec) = -(95\text{cm})\left(\frac{\pi}{3}\frac{\text{rad}}{\text{s}}\right)\sin\left(\frac{5\pi}{6}\right) = -50\text{cm/s}$

$\ddot{x}(2\sec) = -(95\text{cm})\left(\frac{\pi}{3}\frac{\text{rad}}{\text{s}}\right)^2\cos\left(\frac{5\pi}{6}\right) = 90\text{cm/s}^2$