We finished last week with a calculation of $2 \rightarrow 2$ scattering in $\phi^3$ theory.

\[ iM_{2 \rightarrow 2} = \frac{1}{s} \uparrow \uparrow \uparrow + \frac{1}{t} \uparrow \uparrow \uparrow + \frac{1}{u} \uparrow \uparrow \uparrow = -ig^2 \left[ \frac{1}{s-m^2+i\epsilon} + \frac{1}{t-m^2+i\epsilon} + \frac{1}{u-m^2+i\epsilon} \right] \]

\[ \Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2E_{cm}} |M|^2 = \frac{g^4}{16\pi^2E_{cm}} \left[ \frac{1}{s} \right]^2 \]

$s, t, u$ are Mandelstam Variables \[ 2 \rightarrow 2 + 1 \rightarrow 3 \]

Another Example \rightarrow Derivative couplings + Equivalence after integration by parts

First = $\lambda \phi_1(\partial_m \phi_2)(\partial_n \phi_3)$ (derivatives will pick up factors of momenta in LSZ formula

\[ -i \partial^\mu \text{ for momenta leaving vertex } \uparrow \uparrow \uparrow \]

\[ -i \partial\mu \text{ " entering " } \uparrow \uparrow \uparrow \]

Example \[ \Sigma_2^1 \Sigma_3^2 \rightarrow \Sigma_2^1 \Sigma_3^2 \]

\[ 2 \rightarrow 2 + 2 \rightarrow 2 \]

\[ \Rightarrow iM = -i\lambda^2 \left( p \cdot k \right) \left( p_1' \cdot k \right) = -i\lambda^2 \left( m_1^2 + p_1 \cdot p_2 \right) \left( m_2^2 + p_1' \cdot p_2' \right) \]

\[ \frac{1}{h^2 + m_2^2} \]

Now Integrate by parts and recalculate:

\[ \Sigma_{int} = -\lambda \phi_3(\partial_m \phi_1)(\partial_n \phi_2) - \lambda \phi_3 \phi_1 \partial \phi_2 \text{ for } \phi_2 \text{ on-shell} \]

\[ -i\lambda(\partial p_1 \cdot \partial p_2) = i\lambda p_1 \cdot p_2 \]

\[ -i\lambda(\partial p_2) = -i\lambda p_2^2 = -i\lambda m_2^2 \]

\[ iM = i(i\lambda)^2 \left[ p_1 \cdot p_2 \left( p_1' \cdot p_2' \right) + (p_1 \cdot p_2) m_2^2 + m_2^2 \left( p_1' \cdot p_2' \right) + m_2^4 \right] \]

\[ \frac{1}{h^2 + m_2^2} \]

just expansion of what we had above.
Reap we now have methodology for \((\text{GFT} \rightarrow \langle S | \mathcal{L} \rangle | \Omega, \Pi)\) \\
Theory \rightarrow \text{Experiment}

It is now time to properly explore the LHS of this "equation" \(\mathcal{L}\), before we tackle this "equation". What theories might we consider that satisfy our symmetries? Fundamental Thy \(\leftrightarrow\) Poincaré Invariance

Representations of Lorentz group \& spin

Discussion to follow focuses on Spin-1/2 integral spin

Back to discussing representations of Lorentz group!

\(\rightarrow\) Building blocks of theory of fund. physics are fields living in representations of Lorentz group + translations

\(\rightarrow\) Particles labeled by spin + mass/momentum + other stuff (i.e. charge)

\[\text{Changes under } L, T, J\]

Transformation of states: \(|\psi\rangle \rightarrow \hat{\mathcal{P}} |\psi\rangle\)

\(\rightarrow\) Hilbert space typically organizes into groups of states that mix amongst each other under \(\hat{\mathcal{P}}\) transforms \(\rightarrow\) If no further subsets among one of those groups mix only among themselves, representation is irreducible

In order for transformations like \(\hat{\mathcal{P}}\) to be symmetries, require that matrix elements are unchanged:

\[\langle \psi_1 | \psi_2 \rangle \rightarrow \langle \psi_1 | \hat{\mathcal{P}}^+ \mathcal{P} | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle \Rightarrow \hat{\mathcal{P}} \text{ is unitary}\]

\((\text{Aside: another possibility is } \langle \psi_1 | \psi_2 \rangle \rightarrow \langle \psi_1 | \psi_2 \rangle^* \Rightarrow \hat{\mathcal{P}} \text{ or other x-form is anti-unitary})\)

\(\rightarrow\) \(\hat{\mathcal{P}}\) or other x-form is anti-unitary

\(\rightarrow\) Necessity for time-reversal implementation

Particle \(\leftrightarrow\) unitary representation of Poincaré group

\(\rightarrow\) Statement about free theory constraints

\(\ast\) S-matrix is unitary is constraint in general interacting theory
No finite dimensional unitary rep. of Poincaré

- e.g. failure of single particle rel. quantum mechanics

Wigner: reps classified by M=0 mass and spin J=0, \( \frac{1}{2} \), 1, ...

and for J>0 2J+1 indep. states when M ≠ 0
2 indep. states when M = 0

Now compare this with naive quantization of vector potential:

\( A_\mu \): 4-component field \( \Rightarrow 4 \) \( a_\mu, a_\mu^\dagger \)'s \( \Rightarrow 4 \) indep. states!

\( 2 \) photon polarizations agrees with Wigner's counting

Apparent conflict between

\[ \mathcal{G} \neq \mathcal{T} \]

Experiment

Group theory/Quantum mech.

(resolution) = (gauge invariance (vector fields))

(we've implemented) = (diffeomorphism invariance (tensors/gravity))

So why isn't \( A_\mu \) a 4-indep. components not unitary?

Consider state of momentum \( \vec{p} \): \( |\psi_\vec{p}\rangle = a_\mu |e_\mu\rangle \)

\( 4 \)-basis states

\( 0 \) norm + 1 (real?)

\[ \langle \psi_\vec{p}_1 | \psi_\vec{p}_2 \rangle = \sum (a_\mu)^2 \]

\( t \)-not summed w/ minkowski metric!

Not Lorentz invariant

Modify norms: \[ \langle \psi_\vec{p}_1 | \psi_\vec{p}_2 \rangle = g^{\mu\nu} a_\mu a_\nu \]

But now probabilities are messed up:

\[ P(\vec{p}_0) = \langle \psi_\vec{p}_0 | \psi_\vec{p}_0 \rangle^2 \times a_0^2 \rightarrow \cosh^2 (\beta a_0) \]

\( \beta \)-boost

\( \Rightarrow \) can't interpret results as probabilities

1) Lorentz group not compact
2) Boosts: anti-hermitian generators

Not doing QM!
We now embark on building Field theories with an eye on whether they are sensible quantum mechanically.

What do we want? Positive def energy good place to start

Recall \[ E = \sum_n \frac{\partial \phi_n}{\partial \phi_n} \phi_n^2 - L \] energy density

**Spin-0**

\[ \mathcal{L} = \frac{1}{2} (\partial \phi)^2 - \frac{1}{\lambda} m^2 \phi^2 \]

\[ E = \frac{\partial \phi}{\partial \phi} - L = \frac{1}{2} [\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2] \geq 0 \] Easy!

**Spin-1**

3 dof \((2J+1)\) if \(m^2 > 0\) by Wigner's Theorem

Smallest object (field) containing 3 dof or more is an

Under rotation subgroup, \(4 = 3 \oplus 1\)

\[ \uparrow \] vector under rot. \(A_i\)

Might guess

\[ \mathcal{L} = -\frac{1}{2} (\partial A_\mu)(\partial \nu A^\mu) + \frac{1}{4} m^2 \phi^2 \]

which gives same form as KG scalar field:

\[(0+m^2) A_\mu = 0\]

**Problem 1**

Just 4 massive scalar fields \(4 = 1 \oplus 1 \oplus 1 \oplus 1\)

**Problem 2**

\[ E = -\frac{1}{2} \left[ A_0^2 + (\nabla A_0)^2 + m^2 A_0^2 \right] + \frac{1}{2} \left[ \dot{A}_i^2 + (\nabla A_i)^2 + m^2 A_i^2 \right] \]

not bounded from below!

4 massive scalars, one badly behaved

Can also include term \((\partial \mu A_\mu)^2\) allowed by Lorentz invariance

\[ \mathcal{L} = \frac{1}{2} \left[ a A_\mu \partial A^\mu + b A_\mu \partial \nu A^\nu + m^2 A_\mu A^\mu \right] \]

\[ \text{com} \quad a A_\mu + b \partial \nu A^\nu + m^2 A_\mu = 0 \]

\[ \partial_\mu \left[ \left((a+b) A_\mu + m^2\right) \partial \mu A_\mu = 0 \right] \quad a = -b \Rightarrow \partial \mu A_\mu = 0 \] removes dof! entire rep
\[ a = -b = \beta \]

\[ \Rightarrow \quad \mathcal{L} = \frac{1}{2} A_\mu \partial^\mu A^\mu - \frac{i}{2} A_\mu \partial^\mu A^\nu \partial^\nu A_\mu + m^2 A_\mu^2 
= -\frac{i}{4} F_{\mu \nu}^2 + \frac{1}{2} m^2 A_\mu^2 \]