N complex scalar fields $\phi_a$:

$$\mathcal{L} = \partial_m \overline{\phi}^a \partial^m \phi_a - V(\overline{\phi} \phi)$$

a) First note you can write in terms of $2N$ real fields as

$$\mathcal{L} = \partial_m \overline{\phi} \partial^m \phi - V(\overline{\phi} \phi)$$

In this form, we see $\mathcal{L}$ is invariant under transformations $\hat{R} : \overline{\phi} \to \hat{R} \overline{\phi}$ where $\hat{R}^T \hat{R} = 1$ and $\hat{R}$ is $2N \times 2N$

$\hat{R}$ are $SO(2N)$ matrices $\Rightarrow \mathcal{L}$ is an invariant under $SO(2N)$ symmetry

$$\text{dim} (SO(2N)) = \frac{2N(2N-1)}{2} = 2N^2 - N$$

* You might have guessed $U(N)$, but this has only $N^2$ generators
* Higgs is a complex scalar field, thus the global symmetry of the Higgs potential is $SO(4)$, not $U(2)$. This is an important part of the phenomenology of the Standard Model.

b) First we need to calculate $\delta \phi_a$ under an infinitesimal transformation:

$$\phi \to \hat{R} \phi = \exp[i \alpha_k \hat{\tau}_k] \overline{\phi} \quad (k=1, 2N^2-N)$$

($\hat{\tau}_k$ are generators for $SO(2N) \times \text{fund}$)

$$\delta \overline{\phi} = i \alpha_k \hat{\tau}_k \phi \overline{\phi}$$

Noether Current is

$$j_m = \sum_n \frac{\partial}{\partial \phi^\dagger_m} \delta \phi^\dagger_n - \overline{j}_m$$

$$(\hat{\tau}_k)_{\dagger} = (\hat{\tau}_k \overline{\phi}) \overline{\phi} \hat{\tau}_k \overline{\phi}$$

So there are $2N^2 - N$ currents, one for each generator.
\[ \text{2) a) } x \rightarrow x + \alpha x^\mu (x^\nu) \\]

Translations \[ x^\mu \rightarrow x^\mu \pm \xi^\mu \]

\[ \Delta (\partial_\mu x^\mu) \rightarrow \xi^\nu \partial_\nu (\partial_\mu x^\mu) = \xi^\nu \partial_\nu \left( \frac{\partial x^\mu}{\partial \phi} \partial_\nu \phi + \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \partial_\nu (\partial_\mu \phi) \right) \]

\[ = \xi^\nu \partial_\mu \left[ \frac{\partial x^\mu}{\partial \phi} - \partial_\nu \left( \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \right) \partial_\nu \phi + \partial_\nu \left( \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) \right] \]

and \[ \Delta x^\sigma = \xi^\nu \left[ \frac{\partial x^\sigma}{\partial \phi} \partial_\nu \phi + \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \partial_\mu \phi \right] \]

\[ = \xi^\nu \left[ \left( \frac{\partial x^\sigma}{\partial \phi} - \partial_\nu \left( \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \right) \right) \partial_\nu \phi + \partial_\mu \left( \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) \right] \]

\[ \Rightarrow \partial_\sigma (\Delta x^\sigma) = \xi^\nu \partial_\sigma \left[ \left( \frac{\partial x^\sigma}{\partial \phi} - \partial_\nu \left( \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \right) \right) \partial_\nu \phi + \partial_\mu \left( \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) \right] \]

So we have

\[ 0 = \partial_\sigma (\Delta x^\sigma) - \Delta \left( \partial_\nu x^\nu \right) = \xi^\nu \partial_\mu \left[ \partial_\sigma \left( \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) - \partial_\sigma \left( \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) \right] \]

\[ \Rightarrow \Delta T_{\mu \nu} = \partial_\sigma \left[ \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right] \]

\[ \Rightarrow T_{\mu \nu} = \partial_\sigma \partial_\nu \left[ \frac{\partial x^\sigma}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \frac{\partial x^\mu}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right] \]

\[ \text{to Note this is divergence of } B_{\mu \nu} \text{ with } B_{\mu \nu} \text{ antisymmetric in 1st 2 indices } \Rightarrow \text{ trivially satisfies } \partial_\mu \Delta T_{\mu \nu} = 0 \]
6) Whenever you add the divergence of an antisymmetric tensor to $\delta^m$, the conserved charge is unchanged:

$$ j^m \rightarrow j^m + \partial_\nu B^{0\nu} \text{ with } B^{0\nu} = B^{0\nu} $$

$$ \Rightarrow \Delta Q = \int d^3x \delta \omega^0 = \int d^3x \partial_\nu B^{0\nu} = \int d^3x \partial_i B^{i0} $$

since $B^{00} = 0$ by antisymmetry.

and this is integral of 3 divergence:

$$ \Delta Q = \int d^2\sigma \partial_\nu B^{i0} = \int d^2\sigma n_i B^{i0} = 0 \text{ for } B^{i0}, n_i = 0 $$

on normal to boundary.

\[ T_{\mu \nu} = \frac{\partial F}{\partial \partial_\mu A_\nu} - g_{\mu \nu} F \]

\[ \text{and the first term is not symmetric} \]

we can make it symmetric by adding a divergence of

\[ B_{\mu \nu} \rightarrow \text{any choice quadratic in fields and } \]

\[ \partial_\mu B_{\nu \mu} \text{ derivatives is a } F_{\mu \nu} \text{ tbd coefficient} \]

\[ T_{\mu \nu} - T_{\nu \mu} = F_{\mu \nu} \partial_\nu A^\sigma - F_{\nu \mu} \partial_\nu A^\sigma \]

\[ + a \partial_\nu (F_{\mu \nu A_\nu}) - a \partial_\nu (F_{\nu A_\mu}) \star \text{Now use } \partial_\nu F_{\mu \nu} = 0 \]

\[ = F_{\mu \nu} \partial_\nu A^\sigma - F_{\nu \mu} \partial_\nu A^\sigma + a (F_{\mu \nu} \partial_\nu A_\nu - F_{\nu A_\mu}) \]

\[ \text{take } a = -1 : \quad = F_{\mu \nu} F^\sigma - F_{\nu \mu} F^\sigma = F_{\mu \nu} F^\sigma + F_{\nu \mu} F^\sigma = 0 \]
So now the question is whether we can find an $x_\mu$ that gives this $B_{\mu\nu}$

Write most general $x_\mu$ quadratic in $A_{\mu
u}$:

$$X_\mu = \alpha A_\mu (\partial_\nu A^\nu) + \beta (\partial_\nu A_\mu) A^\nu + \gamma (\partial_\mu A_\nu) A^\nu + \delta \partial_\mu \partial_\nu A_{\mu\nu}$$

Then

$$B_{\mu\nu} = \partial_\nu A^\mu \left[ \frac{\partial X_\mu}{\partial (\partial_\nu A^\mu)} - \frac{\partial X_\mu}{\partial (\partial_\nu A^\mu)} \right]$$

$$= \partial_\nu A^\mu \left[ (\alpha - \beta) (A_\rho g_{\mu\rho} - A_{\mu\rho} g_{\nu\rho}) \right]$$

Note

$$B_{\mu\nu} - B_{\nu\mu} = (\alpha - \beta) (A_\rho F_{\mu\nu} - A_\mu F_{\nu\rho} + A_\nu F_{\mu\rho})$$

$$\Rightarrow$$

$$\Delta T_{[\mu\nu]} = (\alpha - \beta) \left[ (\partial_\rho A_\mu) F_{\nu\rho} + A_\mu \partial_\rho F_{\nu\rho} - A_{\mu\rho} \partial_\nu (\partial_\rho A_\mu) + A_{\nu\rho} \partial_\mu \partial_\rho A_\mu \right]$$

And

$$T_{[\mu\nu]} = F_{\mu\rho} \partial_\nu A_\rho - F_{\nu\rho} \partial_\mu A_\rho = (\partial_\mu A_\rho - \partial_\nu A_\rho) \partial_\nu A_\rho$$

$$= 2[(\partial_\mu A_\rho) \partial_\nu A_\rho - (\partial_\nu A_\rho) \partial_\mu A_\rho]$$

So

$$\Delta T_{[\mu\nu]}$$ appears to contain extra terms that cannot be removed.

Not so surprising that single index object $x_\mu$ cannot generate general $B_{\mu\nu}$ w/ 3 indices.
\[ L = -\frac{1}{2} \Phi \Box \Phi + \frac{1}{2} \mathcal{M}^2 \Phi^2 - \frac{1}{4!} \Phi^4 \]

a) **Eom:** \[ \Box \Phi - \mathcal{M}^2 \Phi + \frac{\lambda}{3!} \Phi^3 = 0 \]

\[ \Phi = \Phi \Rightarrow c \left[ \mathcal{M}^2 - \frac{\lambda}{3!} c^2 \right] = 0 \Rightarrow \]

\[ c = \Phi \Rightarrow \sqrt{\frac{\mathcal{M}^2}{\lambda}} \]

\[ \sqrt{c = 0} = 0, \quad \sqrt{c = \frac{\mathcal{M}^2}{\lambda}} = -\frac{1}{2} \mathcal{M}^2 \left( \frac{\mathcal{M}^2}{\lambda} \right) + \frac{\lambda}{4!} \left( \frac{36 \mathcal{M}^2}{\lambda^2} \right) \]

\[ = -\left( \frac{3}{2} \right) \frac{\mathcal{M}^4}{\lambda} \]

\[ \Rightarrow 2 \text{ solutions with degenerate energy } \Rightarrow 2 \text{ classical ground states} \]

b) \( L \) has \( \Phi \rightarrow -\Phi \) symmetry

\[ \text{Let's say that ground state is } + \sqrt{\frac{\mathcal{M}^2}{\lambda}} \]

\[ \text{under } \Phi \rightarrow -\Phi, \text{ you switch to alternate ground state, } \] \( \Rightarrow \) symmetry is not respected  \( \mathbb{Z}_2 \) is spontaneously broken

c) \( \Phi(x) = c + \pi \Phi(x) \) \[ \text{take } c = \pm \sqrt{\frac{\mathcal{M}^2}{\lambda}} \]

\[ \Rightarrow L = -\frac{1}{2} \left( c + \pi \right) \Box \left( c + \pi \right) + \frac{1}{2} \mathcal{M}^2 \left( (c + \pi)^2 - \frac{\lambda}{4!} (c + \pi)^4 \right) \]

\[ = -\frac{1}{2} \pi \Box \pi \Phi + \sum_n \frac{\pi^n}{n!} \left( \frac{\partial^n}{\partial \Phi^n} \Phi \right)_{\Phi = c} \]

\[ = -\frac{1}{2} \pi \Box \pi \Phi + \frac{3}{2} \frac{\mathcal{M}^4}{\lambda} + \frac{\pi^2}{2} \left( \mathcal{M}^2 - \frac{\lambda}{2} \frac{\mathcal{M}^2}{\lambda} \right) - \frac{\pi^3}{3!} \mathcal{M}^2 - \frac{\pi^4}{4!} \lambda \]

\[ = -\frac{1}{2} \pi \Box \pi \Phi + \frac{3}{2} \frac{\mathcal{M}^4}{\lambda} - \frac{1}{2} \left( \mathcal{M}^2 \right) \pi^2 - \frac{6 \mathcal{M}^2}{3!} \pi^3 - \frac{2 \pi^4}{4!} \lambda \]

\[ \text{Eom} \]

\[ \Box \Phi + \left( 2 \mathcal{M}^2 \right) \Phi + \frac{\mathcal{M}^2}{2} \pi^2 + \frac{\lambda}{3!} \pi^3 \Rightarrow \pi = 0 \text{ is solution, and } \]

\[ \sqrt{c = 0} = \sqrt{c_{\min} \text{ from before}} \]

Now \( \Phi' = -\Phi \Rightarrow c + \pi' = -(c + \pi) \Rightarrow \pi' = -2c - \pi \)
The gravitational field equations are non-linear → thus the mode solutions are non-linearly dispersive. That is,
\[ w_n^2 = \frac{2\pi}{L} n \]

Thus, until we solve for the differential volume of phase space, we cannot solve for \( I(w) = \frac{1}{V} \frac{d}{dw} E(w) \)

\* *\* We don't know the classical prediction! * \* *\*

For a half-classical half quantum E&M, we recall that we would get the Einstein coefficients wrong if we used the wrong equilibrium distribution for \( I(w) \)

\[ \rightarrow \text{Eq } 1.12: \quad I(w) = \frac{A}{B^1 e^{iBw} - B} \]

\( \text{could never be satisfied for any } A, B, B' \) using classical equipartition for EM waves

\( \rightarrow \text{Since we don't know } I_{gso}(w), \text{ can't make a definitive statement} \)

\( \rightarrow \text{There are other arguments that } \frac{1}{2} \text{ classical } \frac{1}{2} \text{ quantum can't \textit{work}} \) \* double slit experiment where grain field surrounding an electron cannot carry amplitude characteristics of electron \( \rightarrow \) Schrödinger's cat style experiments involving blowing up macroscopic objects based on microscopic phenomena.

Each have their difficulties \( \rightarrow \) no possible equivalent to test using style in our Universe

\( \rightarrow \) 

Interpretation: gravity waves sourced by quantum fluctuations during inflation (haven't observed yet!)

Another issue is that spin field and other massive spin-2 fields mix (states in QED) i.e. \( |\hat{p}, J=2 \rangle = \alpha |\hat{p}, J=2, \text{grav} \rangle + \beta |\hat{p}, J=2, \text{QED} \rangle \)

\( \rightarrow \text{can't makes sense of grav. field not quantum} \)
Proton Polarizations

a) $L = -\frac{1}{4} F_{\mu\nu}^2 + J^\mu A_\mu$

\[ \Box A_\mu - \partial_\mu (\partial^\sigma A_\sigma) = J_\mu \]


\[ -k^2 \tilde{A}_\mu + \mu_k k^\sigma \tilde{A}_\sigma = \tilde{J}_\mu \]

\[ -k^2 \Rightarrow [-k^2 g_{\mu\nu} + \mu_k k_\nu] \tilde{A}^\nu = \tilde{J}_\mu \]

Call the inverse of this operator $\tilde{D}_{\nu\mu}^{(k^2)} = [-k^2 g_{\mu\nu} + \mu_k k_\nu]^{-1}$

\[ \Rightarrow \tilde{A}^\nu = \tilde{D}_{\nu\mu}^{(k^2)} \tilde{J}_\mu \]

Note: not actually invertible as 4x4 matrix.

Now \[ F_{\mu\nu}^{(k^2)} = i k_\mu \tilde{A}^\nu - i k_\nu \tilde{A}^\mu = i k_\mu \tilde{D}^{\rho\nu} \tilde{J}_\rho - i k_\nu \tilde{D}^{\rho\mu} \tilde{J}_\rho \]

\[ \Rightarrow \frac{1}{4} (F_{\mu\nu}^{(k^2)})^2 = \frac{1}{4} \int_0 (k^2) \left( k_\mu \tilde{D}^{\nu\rho} - k_\nu \tilde{D}^{\mu\rho} \right) \left( k_\mu \tilde{D}^{\rho\nu} - k_\nu \tilde{D}^{\rho\mu} \right) \tilde{J}_\rho (k) \tilde{J}_\rho (-k) \]

\[ = \frac{1}{4} \int_0 \left[ k^2 \tilde{D}^{\nu\rho} \tilde{D}^{\mu\rho} - 2 k_\mu k_\nu \tilde{D}^{\rho\nu} \tilde{D}^{\mu\rho} \right] \tilde{J}_\rho (k) \tilde{J}_\rho (-k) \]

\[ \Rightarrow L = \frac{1}{2} \int_0 \left( k^2 \tilde{D}^{\nu\rho} \tilde{D}^{\mu\rho} \tilde{J}_\rho (k) \tilde{J}_\rho (-k) \right) \]

Since \( k^2 \tilde{J}_\rho = 0 \)

\[ \tilde{D}^{\nu\rho} = -\frac{g^{\nu\rho}}{k^2} \]

b) $\partial_\mu J^\mu = 0 \Rightarrow i k_\mu \tilde{J}_\mu = 0 \Rightarrow \text{for } k_\mu = (0, 0, 0, \omega) \text{ have } \omega \tilde{J}_0 - k \tilde{J}_1 = 0$

\[ \tilde{J}_1 = \frac{\omega}{k} \tilde{J}_0 \]

c) $J_\sigma D^{\sigma\rho} \tilde{J}_\rho = -J_\sigma g^{\sigma\rho} \tilde{J}_\rho = -J_0^2 + J_1^2 + J_2^2 + J_3^2$

\[ = -\frac{J_0^2 (1 - \frac{(\omega^2)}{k^2}) + J_2^2 + J_3^2}{(\omega^2 - k^2)} = \frac{J_0^2 + J_2^2 + J_3^2}{k^2} \frac{1}{\omega^2 - k^2} \]
d) we can write \( J_0 \Delta \Phi J_\rho \) as \( \frac{(w^2 - k^2) J_0^2 + J_1^2 + J_2^2}{w^2(w^2 - k^2)} \)

→ If a term has no time derivatives then the relevant equation of motion will have no time derivatives so the propagation of the effect of a source over some distance cannot depend on time. This is the definition of instantaneous.

→ The inverse FT of the \( J_0 \) term has a \( \delta(t - t') \) \( \Rightarrow \) instantaneous

→ As \( w \to k \) for photons \( J_0 \) falls out (an iE in the denominator will prevent the whole thing from vanishing)

and thus there are no dynamics for \( J_0 \) for on-shell photons

⇒ 2 causally propagating degrees of freedom: \( J_1 + J_2 \)

e) → d) cont'd

It only looks instantaneous, btw, if you focus on the \( J_0 \) term alone. The other terms add corrections that enforce causality.