Cross Sections and Decay Rates

Before beginning a calculation of quantum mechanical matrix elements, it is important to understand the observables that depend on such amplitudes, and how to translate matrix elements into things we observe like cross sections or decay rates. In scattering processes, two states interact in some manner, producing some distinct final state, and in decay processes, a single particle state evolves into a multi-particle state via decay. As in non-relativistic quantum mechanics, the things we can measure depend on squared amplitudes, or probabilities. That is, the probability for a system to evolve from a state \( |i, t_i \rangle \) at time \( t_i \) to a final state \( |f, t_f \rangle \) at time \( t_f \) is given by

\[
P_{fi} \sim |\langle i, t_i | f, t_f \rangle|^2.
\]

This is no different in quantum field theory, where we have a continuum of possible initial and final states, except that in this case matrix elements generate probability distributions.

The above matrix element that we refer to is associated with the Schroedinger picture of quantum mechanics. In our discussion, we will be primarily employing the Heisenberg formalism, where the time dependence is instead assigned to the operators of the theory. In this case, the operator that encodes the time evolution of initial momentum eigenstates into final momentum eigenstates is referred to as the \( S \)-matrix:

\[
\langle f | S | i \rangle_{\text{Heisenberg}} = \langle f; \infty | i; -\infty \rangle_{\text{Schroedinger}} \quad (143)
\]

A key assumption about the \( S \)-matrix is that the initial and final states are well-separated in time from any interactions that occurred between the fields. At the early and late times, the states are presumed to be free of non-trivial interactions that contribute to changes in the momenta.\(^{44}\)

Colliders have been the tool of choice for exploring fundamental particles and their interactions, and the canonical method of describing collisions and their outcomes has been in terms of cross sections. Classical mechanics provides a notion of a cross section by considering the collisions of a beam of particles with an object that lies in their path. If the beam of particles intersects with the object, than any

\(^{44}\) In somewhat rough terms, the \( S \)-matrix is the momentum space analog of spatial correlation functions (which you may be more familiar with from condensed matter physics). Indeed, we will shortly see that the \( S \)-matrix corresponds to the space of singularities in the complex structure of spatial correlation functions.
particles intersecting with the area subtended by the object perpendicular to the path of the particles will be scattered. Thus the fraction of particles scattered with be in proportion to this cross sectional area.

A generalization of this concept that shares the same units with this classical notion of scattering probability is to simply take the ratio of the number of particles scattered to the number in the beam per unit area that pass the object:

\[
\sigma = \frac{\text{number of particles scattered}}{\text{time} \times \text{number density in the beam} \times \text{velocity of the beam}} = \frac{N}{t\Phi}
\]

where \(\Phi\) is the flux.

Note that the denominator is highly dependent on the details of the experiment performed, although the cross section itself is independent of such details, and is a universal property of the particles involved and their interactions.

Another observable is the differential cross section, which allows a distinction of the “shape” of the particles being collided. The differential cross section is the derivative of the cross section with respect to solid angle: \(d\sigma/\Omega(\theta, \phi)\). In a classical example, a triangle will have only two scattering angles, while a sphere generates a continuum.

In particle physics or condensed matter, where new types of particles are being created in collisions, there will be different differential cross sections corresponding to each of the types of final states possible.

In quantum field theory, the notion of area is process dependent, and is related to a probability distribution, however the notion of a cross section is still a useful way of parametrizing the strengths and types of interactions among fields at various energy scales. In terms of differential probabilities associated with the S-matrix of a quantum field theory, the differential cross section goes like

\[
d\sigma = \frac{1}{t\Phi} dP
\]

Note that \(dP\) is differential in the kinematics of the initial and final states and that the flux \(\Phi\) is normalized as though the beam has only one particle in it.

The expected number of scattering events is going to depend on the intensity of the “beams” of the experiment, as well as how long the experiment is run for. These properties of the experiment are combined into a quantity referred to as the integrated luminosity, \(L\), defined by:

\[
dN = Ld\sigma
\]

That is, the number of events in a certain window of momenta/particle type is equal to the luminosity multiplying the differential cross section.
Cross sections from the S-matrix

As we have discussed before, we consider only $2 \rightarrow n$ and $1 \rightarrow n$ processes. Others are possible but are highly suppressed in practice. For now, we focus on initial states with two particles with distinct momenta, and on processes taking those two particles with momentum $p_1$ and $p_2$ to a final state with a set of particles with various momenta:

$$p_1 + p_2 \rightarrow \{ p_j \}$$  \hspace{1cm} (147)

The flux is given by $\Phi = \frac{|\mathbf{v}|}{V}$, where $V$ is the total volume of the system (this will eventually drop out in taking the continuum limit). In terms of relative velocity of the colliding particles, we have $\Phi = \frac{\vec{v}_1 - \vec{v}_2}{V}$. Thus

$$d\sigma = \frac{V}{t} \frac{1}{|\vec{v}_1 - \vec{v}_2|} dP$$  \hspace{1cm} (148)

In terms of the $S$ matrix elements, we express $dP$ as

$$dP = \frac{|\langle f | S | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\Pi$$  \hspace{1cm} (149)

where $d\Pi$ is a differential element of the phase space for the final state. The normalizations of the initial and final states are singular, but will serve to compensate for the extra volume factor (which we want to eventually take to infinity).

We express the phase space factor in terms of the final state momenta as

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j$$  \hspace{1cm} (150)

Note that $\int d\Pi = 1$ as we take the continuum limit.

Remember our normalization of single particle states: $\langle p | k \rangle = (2\pi)^3 (2\omega_p) \delta^{(3)}(\vec{p} - \vec{k})$, meaning that our normalization over states is going to produce a multitude of volume factors: $\langle p | f \rangle \rightarrow 2\omega_p V$.

$$\langle i | i \rangle = 2E_1 V \cdot 2E_2 V \quad \text{and} \quad \langle f | f \rangle = \prod_j 2E_j V$$  \hspace{1cm} (152)

Similarly, we have $(2\pi)^4 \delta^{(4)}(0) = Vt$.

As before, the scattering matrix elements can be expressed as an identity matrix (in a non-interacting theory, this is the only term in the S-matrix, which is entirely the identity matrix):

$$S = 1 + iT$$  \hspace{1cm} (153)

where the operator $T$ is called the transfer matrix, which encodes the contributions due to the non-trivial interactions in the Hamiltonian.
Since we know that the transfer matrix can only have support on processes that obey 4-momentum conservation, we can factor out this delta function from the transfer matrix:

\[ T = (2\pi)^4 \delta^{(4)}(\sum p) \mathcal{M} \]  

(154)

and thus we have

\[ \langle f | \mathcal{S} - 1 | i \rangle = i(2\pi)^4 \delta^{(4)}(\sum p) \langle f | \mathcal{M} | i \rangle \]  

(155)

Now \( dP \) is expressed in terms of the squared matrix element, which will of course involve the square of the above delta function. One of those delta functions serves to multiply the expression by an additional factor of \( \mathcal{V}t \), and the other enforces overall 4-momentum conservation on the scattering process.

Ignoring the contribution of the identity matrix, and using the shorthand \( \mathcal{M} = \langle f | \mathcal{M} | i \rangle \), we then have

\[
\begin{align*}
    d\sigma &= \frac{V}{T} \frac{1}{|\mathcal{V}_{\text{rel}}|} dP = \frac{V^2}{(2E_1V)(2E_2V)} \frac{1}{\prod_j 2E_j V} \mathcal{M}^2 \prod_j V \frac{d^3 p_j}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(\sum p) \\
    &= \frac{1}{2E_12E_2|\mathcal{V}_{\text{rel}}|} \mathcal{M}^2 \left[ \prod_j \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \right] (2\pi)^4 \delta^{(4)}(\sum p).
\end{align*}
\]  

(156)

Note that all factors of the time interval of the interaction and the volume of the space have dropped out. So long as these are large in comparison with the actual time scale and dimensions of the interaction process in some given experiment, this is a valid limit to perform.

The final factor is referred to as the Lorentz invariant phase space:

\[
    d\Pi_{\text{LIPS}} = \left[ \prod_j \frac{d^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \right] (2\pi)^4 \delta^{(4)}(\sum p).
\]  

(157)

which can be demonstrated to be a Lorentz invariant quantity. In fact, the entire differential cross section is a Lorentz invariant quantity, with the prefactor involving a Lorentz invariant function of the initial state, and the squared matrix element being an independently invariant quantity.

**Decay Rates**

We can define the decay rate of a particle by considering a bath of such particles, and the rate of decay is then

\[
    \Gamma = \frac{\# \text{ of decays per unit time}}{\# \text{ of particles present}}
\]  

(158)
We can relate this to probability amplitudes corresponding to $S$-matrix elements (where the amplitudes correspond to probabilities that a one particle state evolves into some $n$-particle state) as

$$d\Gamma = \frac{1}{T} dP$$  \hspace{1cm} (159)$$

In order to relate this to matrix elements you can calculate using the tools of QFT, one follows a derivation that parallels the discussion above, but for a $1 \rightarrow n$ process:

$$d\Gamma = \frac{1}{T} \frac{\langle f | S | i \rangle^2}{\langle f | f \rangle \langle i | i \rangle} d\Pi$$  \hspace{1cm} (160)$$

where again, we have

$$\langle i | i \rangle = (2\pi)^3 2E_i \delta^{(3)}(\vec{p}_i - \vec{\tilde{p}}_i)$$

$$\rightarrow 2E_i V$$

and

$$\langle f | f \rangle \rightarrow \prod_j (2E_j V)$$

and

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j$$

and

$$|\langle f | S | i \rangle|^2 = \left| (2\pi)^4 \delta^{(4)}(\sum_j p_j)M \right|^2 \rightarrow VT (2\pi)^4 \delta^{(4)}(\sum_j p_j) |M|^2$$  \hspace{1cm} (161)$$

Combining everything, and utilizing the above definition for a differential element of Lorentz invariant phase space, we get a simple expression for decay rates that is independent of $T$ and $V$:

$$d\Gamma = \frac{|M|^2}{2E_i} d\Pi_{LIPS}$$  \hspace{1cm} (162)$$

**Example: $2 \rightarrow 2$ scattering**

Let us work out the example of a cross section for a $2 \rightarrow 2$ scattering process.

$$\text{DIAGRAM HERE}$$  \hspace{1cm} (163)$$

We will not calculate $M$, as we need a theory for that. Instead, let us work out the kinematics using whatever constraints we can impose while still knowing nothing about the specifics of $M$. In this case, all of our work involves working out a simplified expression for the Lorentz invariant phase space factor:

$$d\Pi_{LIPS}^2 \rightarrow 2 = (2\pi)^4 \delta^{(4)}(\sum_j p_j) \frac{d^3 p_3}{(2\pi)^3} \frac{1}{2E_3} \frac{d^3 p_4}{(2\pi)^3} \frac{1}{2E_4}$$  \hspace{1cm} (164)$$

Since any physical thing we measure will involve an integral over some finite amount of this phase space, let us proceed to eliminate
some of the delta functions. For example, we could integrate over all 3 components of $\vec{p}_4$:

$$d\Gamma_{\text{LIPS}}^{2\to2} \to \frac{1}{16\pi^2} \frac{1}{E_3 E_4} d^3 p_3 \delta(E_{\text{CM}} - E_3 - E_4)$$

$$= \frac{1}{16\pi^2} \frac{1}{E_3 E_4} dp_f p_f^2 d\Omega \delta(E_{\text{CM}} - E_3 - E_4)$$

(165)

In the last line, we have taken $|\vec{p}_3| = |\vec{p}_4| = p_f$, as is required by 3-momentum conservation in the CM frame, where $p_{\text{tot}} = 0$. Now the delta function restricts $p_f$ to the solution of the function appearing inside it, where we require that

$$E_{\text{CM}} = \sqrt{m_3^2 + p_f^2} + \sqrt{m_4^2 + p_f^2}$$

(166)

but this gives an overall factor of $\frac{E_3 E_4}{p_f(E_3 + E_4)} = \frac{E_3 E_4}{p_f E_{\text{CM}}}$ due to the Jacobian of the transformation to $E$-coordinates. Including this factor, and integrating over $p_f$ yields:

$$d\Gamma_{\text{LIPS}}^{2\to2} \to \frac{1}{16\pi^2} \frac{p_f}{E_{\text{CM}}} d\Omega$$

(167)

Now we can go ahead and plug this into our expression for the general cross section formula. It will serve us well to simplify the term involving the relative velocities, so we can first write

$$|\vec{v}_1 - \vec{v}_2| = \left| \frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2} \right| = \frac{E_{\text{CM}}}{E_1 E_2}$$

(168)

With this substitution, we finally have

$$d\sigma = \frac{1}{64\pi^2 E_{\text{CM}}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{M}|^2 d\Omega$$

(169)

Schwartz then goes to do some “guesswork” to first work out the cross section in the non-relativistic limit for elastic scattering of an electron off of a proton. We can get the result using dimensional analysis and guesswork about the form of field theory interactions of the photon with the electron and proton using the electric current we read off from Noether’s theorem. In the non-relativistic limit, the time derivative of the field is simply oscillatory with the mass serving as the frequency, and only the $A_0$ term contributes. From this, the interaction part of the Hamiltonian will contain a factor of $2em_p$ or $2em_e$ (the factor of two is due to the fact that the current contains the hermitian conjugate in its definition.

So we expect the matrix element for scattering will involve a photon exchanged between the proton and electron, with a propagator relating the fields at the positions of the respective particles due to

\footnote{There is only a physical solution to this equation if $E_{\text{CM}} \geq m_3 + m_4$ - the cross section will vanish if there is no solution (and no available phase space for the process to occur). Note that this only happens at the level of the Lorentz invariant phase space integral since we pulled out the delta function enforcing 4D momentum conservation. The amplitude $\mathcal{M}$ can be non-zero even in the non-physical region of final state momenta.}
the other charged particle, and the propagator should provide a factor of $\frac{1}{k^2}$.

Altogether, we expect

$$\mathcal{M} \approx -2em_p \frac{1}{k^2} 2em_e$$  \hspace{1cm} (170)

which we can now plug into the expressions for $2 \rightarrow 2$ scattering that we derived above:

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \frac{1}{m_p^2} |\mathcal{M}|^2 = \frac{e^4m_e^2}{4\pi^2} \frac{1}{k^4}$$  \hspace{1cm} (171)

This result agrees with the Born approximation for the scattering process, as you would derive it in non-relativistic quantum mechanics.