Old Fashioned Perturbation Theory

We will spend only a short time reviewing old fashioned perturbation theory and exploring its application in quantum field theory. The essentials in QFT are the same as they were in non-relativistic quantum mechanics, where we split the Hamiltonian into two parts - a "free" solvable Hamiltonian, $H_0$ and an interacting part $H_{\text{int}}$ (where the interactions are small in some sense, relative to the interactions). We can then write the full Hamiltonian as

$$H = H_0 + H_{\text{int}}$$

(134)

In most of our applications, $H_0$ is the Hamiltonian of free non-interacting fields.

Say we have a state $|\psi\rangle$ which is an eigenstate of the full Hamiltonian, $H$ with energy $E$:

$$H |\psi\rangle = E |\psi\rangle$$

(135)

Now say we have an eigenstate of $H_0$, $|\phi\rangle$, with the same energy $E$ (which lies in the same Hilbert space as states corresponding to $H$):

$$H_0 |\phi\rangle = E |\phi\rangle$$

(136)

Trivially, we have equality in the following expression:

$$(E - H_0 + ie) |\psi\rangle = (E - H_0 + ie) |\phi\rangle + H_{\text{int}} |\psi\rangle,$$

(137)

when we take the limit as $e \to 0$. Constructing the inverse of $(E - H_0 + ie)$, then we have

$$|\psi\rangle = |\phi\rangle + [E - H_0 + ie]^{-1} H_{\text{int}} |\psi\rangle,$$

(138)

which is called the Lippmann-Schwinger equation.$^{43}$

We can now write the above equation as

$$|\psi\rangle = [1 - \Pi_{LS} H_{\text{int}}]^{-1} |\phi\rangle$$

$$= [1 + \Pi_{LS} H_{\text{int}} + \Pi_{LS} H_{\text{int}} \Pi_{LS} H_{\text{int}} + \cdots] |\phi\rangle$$

$$= |\phi\rangle + \Pi_{LS} T |\phi\rangle$$

(139)
where in the second line we have assumed that $H_{\text{int}}$ is small, allowing an expansion. In the last line, we defined a transfer matrix, $T$:

$$T \equiv H_{\text{int}} + H_{\text{int}} \Pi_{\text{LS}} H_{\text{int}} + H_{\text{int}} \Pi_{\text{LS}} H_{\text{int}} \Pi_{\text{LS}} H_{\text{int}} + \cdots$$  \hspace{1cm} (140)$$

Now we can compute the projection of eigenstates of $H$ onto the corresponding eigenstates of the unperturbed Hamiltonian, $H_0$:

$$\langle \phi_f | \psi \rangle = \langle \phi_f | 1 + \Pi_{\text{LS}} T | \phi_i \rangle = \langle \phi_f | \Pi_{\text{LS}} T | \phi_i \rangle$$  \hspace{1cm} (141)$$

where in the last line, we have presumed that $|\phi_i\rangle$ and $|\phi_f\rangle$ are distinct (and thus orthogonal) eigenstates of $H_0$ with the same total energy $E$. The insertion of $\Pi_{\text{LS}}$ provides an energy conserving delta function multiplying the matrix element $\langle \phi_f | T | \phi_i \rangle$, so we would like to calculate these matrix elements. By inserting the identity in the form of a sum over eigenstates of $H_0$, we have

$$\langle \phi_f | T | \phi_i \rangle \equiv T_{fi} = (H_{\text{int}})_{fi} + (H_{\text{int}})_{fk} \Pi_{kl} (H_{\text{int}})_{li} + \cdots$$  \hspace{1cm} (142)$$

where we have $\Pi_{kl} = \frac{1}{E - E_i}$. The transition rates can then be determined to arbitrary precision in the perturbative couplings in $H_{\text{int}}$ by computing matrix elements of $H_{\text{int}}$ between eigenstates of the unperturbed Hamiltonian, $H_0$.

If we take into account only the leading term in $H_{\text{int}}$, it is referred to as the **Born approximation**.

This is not the modern method of doing perturbation theory in relativistic quantum mechanics. The technique of Feynman diagrams considerably simplifies the process of calculating at successive orders in the interaction terms, and makes it simpler to understand the many divergences that occur when attempting to follow this prescription. We won’t cover it in lecture, but you might find it instructive to read the rest of Chapter 4 in Schwartz’ text to understand how hard the early pioneers of QFT had things.
Cross Sections and Decay Rates

Before beginning a calculation of quantum mechanical matrix elements, it is important to understand the observables that depend on such amplitudes, and how to translate matrix elements into things we observe like cross sections or decay rates. In scattering processes, two states interact in some manner, producing some distinct final state, and in decay processes, a single particle state evolves into a multi-particle state via decay. As in non-relativistic quantum mechanics, the things we can measure depend on squared amplitudes, or probabilities. That is, the probability for a system to evolve from a state $|i, t_i\rangle$ at time $t_i$ to a final state $|f, t_f\rangle$ at time $t_f$ is given by

$$P_{fi} \sim |\langle i, t_i | f, t_f \rangle|^2.$$  

This is no different in quantum field theory, where we have a continuum of possible initial and final states, except that in this case matrix elements generate probability distributions.

The above matrix element that we refer to is associated with the Schrödinger picture of quantum mechanics. In our discussion, we will be primarily employing the Heisenberg formalism, where the time dependence is instead assigned to the operators of the theory. In this case, the operator that encodes the time evolution of initial momentum eigenstates into final momentum eigenstates is referred to as the $S$-matrix:

$$\langle f | S | i \rangle_{\text{Heisenberg}} = \langle f; \infty | i; -\infty \rangle_{\text{Schrödinger}}$$ (143)

A key assumption about the $S$-matrix is that the initial and final states are well-separated in time from any interactions that occurred between the fields. At the early and late times, the states are presumed to be free of non-trivial interactions that contribute to changes in the momenta.\(^{44}\)

Colliders have been the tool of choice for exploring fundamental particles and their interactions, and the canonical method of describing collisions and their outcomes has been in terms of cross sections. Classical mechanics provides a notion of a cross section by considering the collisions of a beam of particles with an object that lies in their path. If the beam of particles intersects with the object, than any

\(^{44}\) In somewhat rough terms, the $S$-matrix is the momentum space analog of spatial correlation functions (which you may be more familiar with from condensed matter physics). Indeed, we will shortly see that the $S$-matrix corresponds to the space of singularities in the complex structure of spatial correlation functions.
particles intersecting with the area subtended by the object perpendicular to the path of the particles will be scattered. Thus the fraction of particles scattered with be in proportion to this cross sectional area.

A generalization of this concept that shares the same units with this classical notion of scattering probability is to simply take the ratio of the number of particles scattered to the number in the beam per unit area that pass the object:

$$\sigma \equiv \frac{\text{number of particles scattered}}{\text{time} \times \text{number density in the beam} \times \text{velocity of the beam}} = \frac{N}{t\Phi}$$

where $\Phi$ is the flux.

Note that the denominator is highly dependent on the details of the experiment performed, although the cross section itself is independent of such details, and is a universal property of the particles involved and their interactions.

Another observable is the differential cross section, which allows a distinction of the “shape” of the particles being collided. The differential cross section is the derivative of the cross section with respect to solid angle: $\frac{d\sigma}{d\Omega}(\theta, \phi)$. In a classical example, a triangle will have only two scattering angles, while a sphere generates a continuum.

In particle physics or condensed matter, where new types of particles are being created in collisions, there will be different differential cross sections corresponding to each of the types of final states possible.

In quantum field theory, the notion of area is process dependent, and is related to a probability distribution, however the notion of a cross section is still a useful way of parametrizing the strengths and types of interactions among fields at various energy scales. In terms of differential probabilities associated with the S-matrix of a quantum field theory, the differential cross section goes like

$$d\sigma = \frac{1}{t\Phi} dP$$

Note that $dP$ is differential in the kinematics of the initial and final states and that the flux $\Phi$ is normalized as though the beam has only one particle in it.

The expected number of scattering events is going to depend on the intensity of the “beams” of the experiment, as well as how long the experiment is run for. These properties of the experiment are combined into a quantity referred to as the integrated luminosity, $L$, defined by:

$$dN = Ld\sigma$$

That is, the number of events in a certain window of momenta/particle type is equal to the luminosity multiplying the differential cross section.

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45 Examples include beam-on-target, beam-on-beam, free streaming particle on free streaming particle (i.e. thermal or astrophysical scattering).

46 I.e. $\sigma(\mu^+ \mu^- \rightarrow h \rightarrow \gamma\gamma)$ vs $\sigma(\mu^+ \mu^- \rightarrow Z^0 \rightarrow e^+e^-)$.

47 In some experiments, the initial state is narrowly spread in momentum, but this is not the case in hadron colliders, in which the colliding particles are the constituents of the proton - quarks and gluons. In such cases the cross sections for $pp$ collisions will involve integrals over the momentum distributions of the quarks and gluons (referred to as parton distribution functions).

48 Beam intensity may be related to densities of fixed targets, luminosity of an actual beam, velocities and densities of dark matter particles in the galaxy, thermal phonon distribution in matter, etc, etc,...
Cross sections from the S-matrix

As we have discussed before, we consider only $2 \rightarrow n$ and $1 \rightarrow n$ processes. Others are possible but are highly suppressed in practice. For now, we focus on initial states with two particles with distinct momenta, and on processes taking those two particles with momentum $p_1$ and $p_2$ to a final state with a set of particles with various momenta:

$$p_1 + p_2 \rightarrow \{p_j\} \quad (147)$$

The flux is given by $\Phi = \frac{|\mathbf{v}|}{V}$, where $V$ is the total volume of the system (this will eventually drop out in taking the continuum limit). In terms of relative velocity of the colliding particles, we have $\Phi = \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{V}$. Thus

$$d\sigma = \frac{V}{t} \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|} dP \quad (148)$$

In terms of the $S$ matrix elements, we express $dP$ as

$$dP = \frac{|\langle f | S | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\Pi \quad (149)$$

where $d\Pi$ is a differential element of the phase space for the final state. The normalizations of the initial and final states are singular, but will serve to compensate for the extra volume factor (which we want to eventually take to infinity).

We express the phase space factor in terms of the final state momenta as

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3 p_j \quad (150)$$

Note that $\int d\Pi = 1$ as we take the continuum limit.

Remember our normalization of single particle states: $\langle p | k \rangle = (2\pi)^3 (2\omega_p) \delta^{(3)}(\mathbf{p} - \mathbf{k})$, meaning that our normalization over states is going to produce a multitude of volume factors: $\langle p | p \rangle \rightarrow 2\omega_p V$.$^{49}$

$$\langle i | i \rangle = 2E_1 V \cdot 2E_2 V \quad \text{and} \quad \langle f | f \rangle = \prod_j 2E_j V \quad (152)$$

Similarly, we have $(2\pi)^4 \delta^{(4)}(0) = Vt$.

As before, the scattering matrix elements can be expressed as an identity matrix (in a non-interacting theory, this is the only term in the S-matrix, which is entirely the identity matrix):

$$S = 1 + iT \quad (153)$$

where the operator $T$ is called the transfer matrix, which encodes the contributions due to the non-trivial interactions in the Hamiltonian.
Since we know that the transfer matrix can only have support on processes that obey 4-momentum conservation, we can factor out this delta function from the transfer matrix:

\[ T = (2\pi)^4 \delta^{(4)}(\sum p) \mathcal{M} \]  

and thus we have

\[ \langle f | \mathcal{S} - 1 | i \rangle = i(2\pi)^4 \delta^{(4)}(\sum p) \langle f | \mathcal{M} | i \rangle \]  

Now \( d\mathcal{P} \) is expressed in terms of the squared matrix element, which will of course involve the square of the above delta function. One of those delta functions serves to multiply the expression by an additional factor of \( V_\text{t} \), and the other enforces overall 4-momentum conservation on the scattering process.

Ignoring the contribution of the identity matrix, and using the shorthand \( \mathcal{M} = \langle f | \mathcal{M} | i \rangle \), we then have

\[
\frac{d\sigma}{\mathcal{P}} = \frac{V}{|\mathcal{E}_\text{rel}|} \frac{1}{(2E_1V)(2E_2V)} \frac{V^2}{\prod_j 2E_j V} |\mathcal{M}|^2 \prod_j \frac{d^3 p_j}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(\sum p) \]

\[
= \frac{1}{2E_1 2E_2 |\mathcal{E}_\text{rel}|} |\mathcal{M}|^2 \prod_j \frac{d^3 p_j}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(\sum p). \]  

Note that all factors of the time interval of the interaction and the volume of the space have dropped out. So long as these are large in comparison with the actual time scale and dimensions of the interaction process in some given experiment, this is a valid limit to perform.

The final factor is referred to as the Lorentz invariant phase space:

\[ d\Pi_{\text{LIPS}} = \prod_j \frac{d^3 p_j}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(\sum p). \]  

which can be demonstrated to be a Lorentz invariant quantity. In fact, the entire differential cross section is a Lorentz invariant quantity, with the prefactor involving a Lorentz invariant function of the initial state, and the squared matrix element being an independently invariant quantity.