where the equations of motion for $\phi$,
\[
(\Box + m^2)\phi^{(\ast)} = 0,
\]
have been enforced in the last equality.

**Noether’s Theorem:**
For a classical field theory with an action that is invariant under some continuous transformation, there is an associated current (and thus total charge) that is conserved when the equations of motion are satisfied.

Noether’s Theorem applies when:

- The symmetries are continuous, and the transformation parameters $a_i$ can be taken infinitesimally small
- The equations of motion are satisfied

Note that it applies when the parameters are local transformations as well - when the $a_i$ are functions of $x$.

**Example: Translation invariance and $T^{\mu\nu}$**

An important example of continuous symmetries are those associated with the region in which our fields “live.” Translation invariance in space and time are part of the symmetries of full Poincaré invariance. Let us consider the change in a Lagrangian due to a translation in spacetime coordinates $x^\mu \rightarrow x^\mu - \xi^\mu$. Under such a translation, the shift in a scalar field $\phi$ is $\phi(x) \rightarrow \phi(x + \xi) \approx \phi(x) + \xi^\mu \partial_\mu \phi(x)$, where the last term is simply the Taylor expansion. Note that there are 4 transformation parameters, one for each Minkowski coordinate. We can write the transformation $\xi^\mu = a_i e^\mu_i$, where $i = 0, \cdots, 3$ and the $e^\mu_i$ span a basis for Minkowski space. Thus $\delta_i \phi = e^\mu_i \partial_\mu \phi$.

Also note that the Lagrangian is not invariant, since $\mathcal{L}$ is also a scalar field: $\delta \mathcal{L} = \xi^\mu \partial_\mu \mathcal{L}$, so we have a nonzero $\partial_\mu \mathcal{J}^\mu_i = \partial_\mu (e^\mu_i \mathcal{L})$. The remaining part of the current is
\[
j^{\mu}_i = \sum_n \left[ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} e^\mu_i \partial_\mu \phi_n \right] - e^\mu_i \mathcal{L} = e^\mu_i \mathcal{S}_{\rho\nu} \left\{ \sum_n \left[ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} \partial_\nu \phi_n \right] - g^{\mu\nu} \mathcal{L} \right\}
\]

The term in brackets on the right we identify as the stress energy tensor:
\[
T^{\mu\nu} = \sum_n \left[ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_n} \partial_\nu \phi_n \right] - g^{\mu\nu} \mathcal{L}
\]

By Noether’s theorem, the stress energy tensor is conserved: $\partial_\mu T^{\mu\nu} = 0$. Note that this corresponds to 4 Noether currents - one each for the

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\* This is of great importance when we go on to discuss gauge theories, where vector fields can couple only to conserved currents. If you try to quantize the theory otherwise, there are issues with unitarity.

\* If $\xi$ is a function of coordinates, then this is a diffeomorphism transformation. Here we take $\xi$ to be a global transformation, or a constant shift.

\* This definition of the stress-energy tensor is not necessarily symmetric, but can be made so through adding terms as is done in the definition of the Belinfante-Rosenfeld stress energy tensor (which is the one that couples to gravity).
four possible translations. The four conserved “charges” are

\[ Q^\mu = \int d^3x T^{0\mu} \]  

(112)

and they correspond to total energy \( Q^0 \) and 3-momentum \( Q^i \).

We now have an explanation for the rules of conservation of energy and momentum. If physics is the same at any place in Minkowski space, then energy and momentum are conserved.

**Currents and Sources**

Let us make a couple comments about the different definitions of “currents” that we typically employ.

First, currents can be functions of fields associated with a continuous symmetry via Noether’s Theorem.

Second, currents can be configurations of (possibly moving) external charges. By external, we mean that they are fixed functions of time and space, and that they act then as source terms in the equations of motion. I.e., in Maxwell’s equations we have \( \partial_\mu F^{\mu\nu} = j^\nu \), where \( j^\nu \) is a conserved current. If \( j^\nu \) is non-zero, it is a source term for electromagnetic dynamics.

Finally, it is often convenient in interacting theories to employ a shorthand for the set of interactions of a vector field. For example, under electromagnetism there are contributions to the Noether currents that couple to \( A_\mu \) for every single particle type that carries electric charge, and it is convenient to write all these interactions as \( \mathcal{L} \ni -A_\mu j^\mu \).

**Coulomb’s Law**

Before embarking on the quantum mechanics of field theory, let us try a calculation in classical field theory that will serve as a warm-up to its quantized counterpart.

We can derive Coulomb’s Law using techniques in classical field theory (many of which we continue to employ in quantum field theory later). We start with a charge \( e \) at the origin, which corresponds to an external source/current:

\[ j^0 = \rho(x) = e\delta^3(x) \quad \overset{\sim}{j}(x) = 0 \]

(113)

The Lagrangian associated with Maxwell’s equations is

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu j^\mu, \]

(114)

Noting that \( \frac{\partial F_{\mu\nu}}{\partial \phi} = \delta_{\mu\nu} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\nu} \), and \( \frac{\partial^2 F_{\mu\nu}}{\partial \phi^2} = 2 \frac{\partial F_{\mu\nu}}{\partial \phi} \frac{\partial F_{\mu\nu}}{\partial \phi} \), we have for the Euler Lagrange equation:

\[ \partial_\mu F^{\mu\nu} = \Box A^\nu - \partial_\nu (\partial_\mu A^\mu) = j^\nu \]

(115)
Making the gauge choice $\partial_r A^r = 0$ (Lorenz gauge), we have

\[ \Box A^\mu = j^\mu \quad \text{or} \quad A^\mu = \Box^{-1} j^\mu \quad (116) \]

To obtain the second equation, we have inverted the $\Box$ operator. While it is not immediately clear what this means (unless you are well-versed in Fourier analysis), we might intuit that something very non-local is involved, as the inverse of derivative operators is an integral operation. We say that $A^\mu$ is determined by the propagation of the field from some distribution of sources/currents, and we call $\Box^{-1}$ the **propagator** for $A^\mu$.

To understand better what this means, let us solve the general problem in momentum space. Our equation of motion for the Maxwell theory is $\Box A^\mu = j^\mu$. Let us Fourier transform both sides of this equation. Focusing on the LHS first, we find

\[ \Box A^\mu(x) \rightarrow \int d^4x e^{ikx} \Box A^\mu(x) \]
\[ = -k^2 \int d^4x e^{ikx} A^\mu(x) \]
\[ = -k^2 \tilde{A}^\mu(k). \quad (117) \]

The RHS is

\[ \tilde{j}^\mu(k) = \int d^4y e^{iky} j^\mu(y), \quad (118) \]

giving us a momentum space solution for $A$:

\[ \tilde{A}^\mu(k) = -\int d^4y \frac{1}{k^2} e^{iky} j^\mu(y). \quad (119) \]

Performing the inverse FT of both sides to obtain $A^\mu(x)$, we have

\[ A^\mu(x) = \int d^4y \left[ -\int \frac{d^4k}{(2\pi)^4} e^{ik(y-x)} \frac{1}{k^2} \right] j^\mu(y) \quad (120) \]

Let us define $\Pi(x,y)$ as

\[ \Pi(x,y) = \Pi(y,x) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{1}{k^2} \]
\[ \quad \text{giving us} \quad A^\mu(x) = -\int d^4y \Pi(x,y) j^\mu(y). \quad (121) \]

Let us now return to our study of a stationary point charge at the origin. In this case, we have $\Box A^0 = j^0$ (all other components are zero), and our equation for $A^0$ is:

\[ A^0(x) = -\int d^4y \Pi(x,y) j^0(y) \]
\[ = -\int d^4y \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} e^{ik(x-y)} \epsilon^{(3)}(\vec{y}) \]
\[ \quad \text{giving us} \quad A^0(x) = -\int d^4y \Pi(x,y) j^0(y) \quad (122) \]

\[ \text{Note that } \Pi(x,y) \text{ is a Green's function for the box operator:} \]
\[ \Box_x \Pi(x,y) = -\delta^{(4)}(x-y), \quad (122) \]

and thus serves to help define what we mean by $\Box^{-1}$. 

\[ 39 \]
Performing the integral over $y$, the $\bar{y}$ components are set to zero, while the integral over $y_0$ produces a delta function in $k_0$:

$$A^0(x) = e \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k} \cdot \bar{x}}}{k^2}$$

$$= \frac{e}{4\pi^2} \int d|k| \int d\cos \theta e^{i\bar{k}|r| \cos \theta}$$

$$= \frac{e}{8\pi^2} \frac{1}{ir} \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k}$$

$$= \frac{e}{8\pi^2} \frac{1}{ir} \lim_{\delta \to 0} \left[ \int_{-\infty}^{\infty} dk \frac{e^{ikr} - e^{-ikr}}{k + i\delta} \right]$$

The last integral can be performed using contours that close either above or below the real axis with a contour at $\infty$. For positive $r$, we must close the first exponential integral above to get a vanishing contribution from the contour at $\infty$, which misses the pole, and thus gives no contribution. For the other term, we close below, which picks up the pole at $k = -i\delta$, giving a factor $(-2\pi i)(-e^{-\delta r})$ where the factor of $-2\pi i$ comes from the residue theorem associated with the negatively oriented contour that encircles the pole. The result of the integral is then (after taking the $\delta \to 0$ limit):

$$A^0(x) = \frac{e}{4\pi} \frac{1}{r}$$

which is, of course, the value of the electric potential associated with a positive point charge at the origin, and its gradient gives the expression for the electric field of a point charge.

**Classical Perturbation Theory and Green’s Functions**

Let us now go non-linear. Most field theories of interest are interacting field theories, by which we mean that the SHO/linear wave approximation breaks down at large field amplitudes (as is the case for most/all real physical systems). It is the phenomenon of non-linearity that makes physics interesting - without non-linearities, wave excitations would simply pass through each other, with no meaningful exchange or evolution of the information they contain. Of course this aspect of physics also makes prediction difficult, as non-linear equations are difficult or impossible to study analytically, and we typically resort to either perturbation theory around the linear regime of the equations, or to numerical simulation. For now, let us explore the first of these.

Electrodynamics is a rather exceptional field theory, in that the equations of motion for the vector potential are linear with the exception of dynamical field contributions to the electromagnetic current.
Another theory with massless fields that has different phenomena is that of gravity. Unlike electromagnetism, in Einstein’s theory gravitational waves interact with each other. However, the non-linearities in gravity are very small, and so to a good approximation gravity can be modeled as simply another linear field theory. Let us nonetheless explore the consequences of these non-linearities.

Rather than studying gravity itself, which is very complicated in terms of technical aspects concerning gauge invariance and the many metric degrees of freedom, we consider a toy scalar field theory model (much like we did initially to study electrodynamics).

The Lagrangian we consider is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu h)^2 + \frac{1}{3} \lambda h^3 + Jh.$$  \hfill (127)

If we ignore the $h^3$ term, this Lagrangian leads to a massless Klein-Gordan field equation (which is linear in $h$) for $h$ coupled to a source/current $J$: $\Box h = J$. The $h^3$ term, however, leads to a quadratic term in the field equations:

$$\Box h - \lambda h^2 = J$$  \hfill (128)

If we take $\lambda$ to be small, then we can study solutions for $h$ using perturbation theory. Let us define a function $h_0$ that satisfies the equations in the limit $\lambda = 0$: $h_0 = \frac{1}{\Box} J$, where the RHS can be defined using the Fourier transform as before, for some given source configuration:

$$h_0(x) = - \int d^4y \Pi(x,y) J(y).$$  \hfill (129)

Now let us presume a solution of the form $h = h_0 + h_1 + \cdots$, where $h_1 \sim O(\lambda)$. Plugging this “full solution” into the equations of motion, we have:

$$\Box(h_0 + h_1) - \lambda (h_0 + h_1)^2 = J.$$  \hfill (130)

Now we expand in small $\lambda$, keeping only terms up to linear order in $\lambda$, and using the definition of $h_0$ to eliminate the source term:

$$\Box h_1 - \lambda h_0^2 \approx 0$$  \hfill (131)

This equation is sufficient to determine the $\lambda^1$ term in a perturbative expansion of $h = h_0 + h_1 + \cdots$. We can now finally express $h_1$ in terms of our solution to $h_0$:

$$h_1 = \lambda \frac{1}{\Box} (h_0^2) = - \lambda \int d^4y \Pi(x,y) h_0^2(y)$$
$$= - \lambda \int d^4y \Pi(x,y) \left[ - \int d^4w \Pi(w,y) J(w) \right] \left[ - \int d^4z \Pi(z,y) J(z) \right]$$
$$= - \lambda \int d^4y \int d^4w \int d^4z \Pi(x,y) \Pi(y,w) \Pi(y,z) J(w) J(z)$$  \hfill (132)
Now we have a solution to the equations of motion that is valid up to order $\lambda$:

$$h(x) = -\int d^4y \Pi(x, y) J(y) - \lambda \int d^4y \Pi(x, y) \left[ \int d^4w \Pi(y, w) J(w) \right] + \mathcal{O}(\lambda^2)$$

There is a diagrammatic way to represent these formulae - Feynman Diagrams! Feynman graphs are not particular to quantum field theory, and appear already in classical field theory. A key difference that will emerge is that in quantum mechanics, non-vanishing irreducible fluctuations of the field can effect the propagation from sources. These quantum fluctuations act as intermediate sources that contribute through self interactions to the field value at some position.

Working to arbitrary precision in Lambda is straightforward, and involves the brute force application of the following Feynman Rules:

1. Draw a point $x$ where you wish to know the field value, and a line to a new point $x_1$.

2. At point $x_1$, you can put either a source, and end the process, or you can insert an interaction term. Depending on the order of the interaction $h^3, h^4, ...(if you use an interaction term) you add $n-1$ more lines splitting off from point $x_1$, going to points $x_2, x_3$, etc.

3. Iterate this process up to the order that you wish to work in perturbation theory. Each vertex counts as one power of the small perturbation $\lambda$.

4. The value $h(x)$ is then obtained by integrating over all points but $x$, with propagators $\Pi(x, y)$ representing lines, and sources $J(x_i)$ being functions of the configuration of masses/charges/etc, that create the field.