Introduction

In our efforts (now ~ 1 century old) to unify quantum mechanics and relativity, it was learned that another unification was needed. Namely, the field description of electromagnetism was generalized to include matter as well. Without this conceptual leap, earlier attempts to construct a consistent theory were plagued by issues with causality and unitarity (in essence conservation of probability) and by an inability to describe processes in which particle number was not conserved. In addition, the extension of the relativistic field paradigm to include matter (i.e. the electron in Dirac’s field equation) predicted the existence of anti-matter, which was discovered 4 years later in experiment by Carl Anderson in 1932.1

As a point of fact, field theories need not be relativistic, and the tenets of quantum field theory are also of extraordinary utility in describing correlations in various condensed matter systems at length scales that are large in comparison with the microscopic molecular constituents (particularly near critical points where long-range order is present).

In this set of notes, we will speak primarily about systems that exhibit full Poincaré invariance, but the reader is expected to appreciate that relativistic theories are only a small sample of the possibilities afforded by the general framework of quantum field theory. Models with less (or more) symmetry are often employed to provide a theoretical framework that describes phenomena in various experimental contexts.

One result of the application of the field paradigm in quantum physics is that the behavior of a theory typically changes as a function of the energy or length scales associated with various processes. This phenomenon goes under the (somewhat unfortunate) name of renormalization group flow. The reader who knows a bit of the history of fundamental physics might associate with this term a rather cumbersome and unappealing subtraction of infinities that is necessary to make concrete predictions for physical processes.

In this set of lectures, we will strive to give a more modern viewpoint of renormalization as it appears in the framework of Wilsonian

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1 Positron Discovery:

A cloud-chamber photo of a positron track. At the center is a lead plate which impedes the motion of the positron, leading to the smaller radius of curvature of the track in the top-portion of the photo.
effective field theory. With this viewpoint, the infinities of quantum field theory represent real physical thresholds where the micro- or macroscopic phenomena undergo some significant change in behavior. One example is the cross-over between the Fermi theory of the weak interactions valid at energies below $\sim 100$ GeV and the high energy description that involves exchange of the electroweak $W^\pm$ and $Z^0$ bosons. Another example occurs in quantum chromodynamics, where the diverging coupling constant associated with the theory of quarks and gluons corresponds to the confinement scale. Below this scale, the theory involves a zoo of interacting mesons and baryons, while above this scale, gluons and quarks give a valid description. Finally, in condensed matter systems, divergences could correspond to the scale at which a field theory description breaks down, and a full characterization of the microscopic constituents becomes necessary (i.e. at the interatomic spacing scale).

Quantum field theory is a beautiful and versatile theoretical toolkit with applications throughout many sub-fields in physics. It is also our current-best framework for describing fundamental phenomena when employed with the particular field content of the Standard Model of particle physics. Despite these successes, it has become clear that it is a limited tool. Quantum field theory and gravity do not play well together except in very limited circumstances that at first glance appear completely unrelated to our own physical world. The Standard Model is beautiful in many ways, but there is no deep guiding principle for selection of the particular field content and the parameters associated with its interactions. What makes the SM field theory better than the many other possibilities afforded by the more general framework? Resolution of these points lies outside the scope of these notes, and indeed the full solution to these problems is the focus of many avenues of ongoing active research.
The First Quantum Field Theory

A classical description of light emerged in 1861 and 1862 with Maxwell’s publication of his field equations for electromagnetic theory. It was eventually recognized that these did not obey the usual Galilean symmetry transformations, and in the later 1800’s and early 1900’s numerous physicists published the correct symmetry transformations. Einstein later showed that these transformations emerged from the simple axioms of special relativity, and Minkowski soon employed the now modern viewpoint of a 3 + 1 dimensional space-time endowed with Poincaré invariance.

The first hints that the classical description of Maxwell’s were to be overturned involved considerations of thermal spectra. Max Planck in 1900 proposed a phenomenologically motivated description of blackbody radiation that required a new constant now known as Planck’s constant, $h = 6.626 \cdot 10^{-34}$ J·s. His work initiated the theory of the non-relativistic quantum mechanics of electrons, with a full description of the quantum theory of light yet to come. Despite this history, it is appropriate to begin with some quantum mechanical aspects of the electromagnetic field.

Blackbody radiation

The equipartition theorem of statistical mechanics states that a system in thermal equilibrium has the energy equally distributed in all available modes. This theorem leads to a good description of the thermal velocities in a hot gas—the Boltmann distribution, but leads to a paradox when the same methods are applied to the spectrum of light from a hot object referred to as the ultraviolet catastrophe.

A blackbody is an idealized concept of an object with no internal structure, and for our purposes, we consider a box filled only with light in thermal equilibrium with itself at some temperature (an electromagnetic version of the hot gas referred to above). For a box of size $L$, the angular frequencies of electromagnetic waves that are
allowed in the box occupy a spectrum given by

\[ \omega_n = \frac{2\pi}{L} |\vec{n}|c \]  

where \( \vec{n} \) are integer valued 3-vectors. Classically, there is no upper or lower limit to how much energy can be placed in each of these possible modes, and the equipartition theorem implies that the emission spectrum of this blackbody should grow in proportion with the volume of available phase space:

\[ I(\omega) = \frac{1}{V} \frac{d}{d\omega} E(\omega) = \text{const} \frac{\omega^2 k_B T}{c^3} \]  

One can in fact determine this expression simply using dimensional analysis and available fundamental constants. This expression is the only one consistent with the fact that intensity must take the form power/volume.

We see that the intensity must grow quadratically with \( \omega \), diverging for large frequencies. This is in conflict with common sense (most energy would be emitted at ultra high gamma rays) and it also conflicts with experiment, where the spectra look far more like a Maxwell Boltzmann distribution with a peak at finite \( \omega \). Happily, the equipartition theorem appears not to apply as naively expected to electromagnetic thermal emission spectra.

Max Planck’s ad hoc toy model for describing the observed spectrum was a proposal for a quantization of the energy in electromagnetic waves according to the formula

\[ E_n = |\vec{p}_n|c = \hbar \omega_n \]  

where \( \hbar \) is the reduced planck constant: \( \hbar \equiv \frac{h}{2\pi} \). ² Since it becomes cumbersome to always keep track of the various factors of \( c \), we will henceforth set \( c = 1 \), with both distance and time carrying the same units (i.e. that of seconds).

Einstein took the leap to considering this toy model a true description of nature, with light made up of massless particles later referred to as photons. This gives a good explanation of the photoelectric effect, and also of Compton scattering via the finite momentum of a given photon.

Of course it also, as Planck calculated, gives the correct description for blackbody radiation. Each mode of frequency \( \omega_n \) can be excited some integer number, \( j \), times giving a total energy in each mode of \( jE_n = j\hbar \omega_n \), and then the Boltzmann weight can be applied to get the expected energy in each mode:

\[ \langle E_n \rangle = \frac{\sum_{j=0}^{\infty} j E_n e^{-jE_n/\beta}}{\sum_{j=0}^{\infty} e^{-jE_n/\beta}} = \frac{\frac{d}{dp} \frac{1}{1 - e^{-\hbar \omega_n/\beta}}} {\frac{1}{1 - e^{-\hbar \omega_n/\beta}}} = \frac{\hbar \omega_n}{e^{\hbar \omega_n/\beta} - 1} \]

One can then take the continuum limit, \( L \to \infty \) going from a box of hot photons to a universe of hot photons, meaning the spacing between modes goes to zero and all energies are possible. The expected
energy in all low energy modes below some frequency \( \omega \) is then

\[
E(\omega) = \int d^3 \vec{n} \frac{\hbar \omega n}{e^{\hbar \omega / T} - 1} = 4\pi \hbar \frac{1}{8\pi} \int_0^\omega d\omega' \frac{\omega'^3}{e^{\hbar \omega' / \beta} - 1} \tag{5}
\]

Finally, we can recalculate the intensity \( I(\omega) \):

\[
I(\omega) = \frac{1}{V} \frac{dE(\omega)}{d\omega} = \frac{\hbar}{\pi^2} \frac{\omega^3}{e^{\hbar \omega / \beta} - 1}
\]

Thus the emission at low \( \omega \) grows as \( \omega^3 \), but this is cut off at energies \( E \sim \beta^{-1} \), and larger energies in the spectrum are damped by the exponential in the denominator.

At this point you should be asking the question of what precisely this process of equilibration entails, of moving from a system with, say, all photons in one energy mode, to a thermal system described by the distribution above. This equilibration corresponds to photons of various energies being created and destroyed as the system moves towards the intensity profile given above.\(^3\) It is quantum field theory that gives a full multi-particle description of this equilibration process that must violate particle number conservation.

**Spontaneous Emission**

There are processes which contain the characteristics we describe above - where photons are created in some isolated system. Such behavior occurs when an excited atom emits light as it moves from one excited state to another state of lower energy. One could also conceive of a system of atoms at equilibrium that are continually emitting and re-absorbing photons and moving from one excitation state to another.

Einstein considered such a system with two energy states \( E_1 \) and \( E_2 \), with number densities \( n_1 \) and \( n_2 \). Consider photon energies given by \( E = \hbar \omega = E_2 - E_1 \). One can define a constant proportional to the probability for emission of a photon of energy \( E \) as the coefficient for spontaneous emission, \( A \). In addition, these photons can stimulate emission from states \( E_2 \), and the probability is proportional to a coefficient of stimulated emission, \( B \), and also to the number of photons of frequency \( \omega \) present. The change in \( n_2 \) due to all emission processes is then given by

\[
dn_2 = -[A + BI(\omega)] n_2 \tag{7}
\]

It is also possible for photons to be absorbed, and cause an increase in the energy level of the atom from \( E_1 \) to \( E_2 \). The associated probability is proportional to the coefficient of absorption \( B' \). Absorption

\(^3\) It also requires an interaction of photons with photons in our idealized photon bath, something that does not occur in pure electromagnetism, but is present once a full relativistic theory of photons, electrons, and positrons is developed.
decreases $n_1$ and increases $n_2$ by an amount $B' I(\omega) n_1$. The total number of atoms is conserved, so we have

$$dn_2 = -dn_1 = -[A + B I(\omega)] n_2 + B' I(\omega) n_1.$$ (8)

Here $I(\omega)$ is some external intensity profile associated with the photons. In equilibrium, it would be given by the profile we calculated above, or we could contrive to create another type of external source of photons.

Equilibrium implies that $dn_1 = dn_2 = 0$, and that the number densities are in proportion to their respective Boltzmann weights:

$$n_1 = N e^{-E_1 \beta} \quad n_2 = N e^{-E_2 \beta}$$ (9)

where $N$ is a constant of proportionality determined by the density of atoms. The equilibrium constraint then leads to the following relation:

$$\left[ B' e^{-E_1 \beta} - B e^{-E_2 \beta} \right] I(\omega) = A e^{-E_2 \beta}$$ (10)

or

$$I(\omega) = \frac{A}{B' e^{-(E_2 - E_1) \beta} - B} = \frac{A}{B' e^{-h \omega \beta} - B}$$ (11)

where However, we know what the equilibrium distribution of photons is from our previous calculation, from which we find:

- $B' = B$
- $A/B = \frac{\hbar}{\pi^2 \omega^3}$

This final result is great, although it is unclear why we needed to use our previous calculation to derive it. The coefficients of emission and absorption have nothing to do with equilibrium dynamics, yet it was necessary for us to call upon it to get the above result.

We note that the coefficients of stimulated emission and absorption can be calculated using standard quantum mechanics, however the ratio $A/B$ requires (at this stage) the input from equilibrium dynamics of photons. A first-principles calculation of $A/B$ had to wait until the advent of quantum field theory.